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Geometry

$$x = x^1 e_1 + x^2 e_2 + \dots + x^n e_n$$

$$\xi = \xi^1 e_1 + \xi^2 e_2 + \dots + \xi^n e_n$$

$$e_i \cdot e_j = \delta^i_j$$

$$j = \frac{\partial x^i}{\partial \xi^j} = \begin{pmatrix} \frac{\partial x^1}{\partial \xi^1} & \frac{\partial x^2}{\partial \xi^1} & \cdots & \frac{\partial x^n}{\partial \xi^1} \\ \vdots & & \ddots & \\ \frac{\partial x^1}{\partial \xi^n} & \cdots & \cdots & \frac{\partial x^n}{\partial \xi^n} \end{pmatrix}$$

$$J = \det(j)$$

$$x^1 = x^1(\xi^1, \xi^2, \dots, \xi^n) \qquad \xi^1 = \xi^1(x^1, \dots, x^n)$$

$$\vdots \qquad \text{similarly} \qquad \vdots$$

$$x^n = x^n(\xi^1, \xi^2, \dots, \xi^n) \qquad \xi^n = \xi^n(x^1, \dots, x^n)$$

$$\therefore\ jj^{-1} = I \qquad \therefore\ \frac{\partial x^i}{\partial \xi^k}\frac{\partial \xi^k}{\partial x^j} = \frac{\partial x^i}{\partial x^j} = \delta^i_j, \text{ similarly } \frac{\partial \xi^i}{\partial x^k}\frac{\partial x^k}{\partial \xi^j} = \delta^i_j$$

$$x(\xi) = x^1(\xi) e_1 + \dots + x^n(\xi) e_n$$

tangential vector:

range of space.

$$x_{\xi^i} = \frac{\partial x^1}{\partial \xi^i} e_1 + \dots + \frac{\partial x^n}{\partial \xi^i} e_n$$

$$\nabla \xi^i = \frac{\partial \xi^i}{\partial x^1} e_1 + \dots + \frac{\partial \xi^i}{\partial x^n} e_n$$

$$\because\ jj^{-1} = I, \quad \therefore\ x_{\xi^i} \cdot \nabla \xi^j = \delta^j_i$$

in 3D, $J = x_{\xi^1} \cdot (x_{\xi^2} \times x_{\xi^3}) \Rightarrow \nabla \xi^i = \frac{1}{J}(x_{\xi^l} \times x_{\xi^m})$

$$\Downarrow$$

$$\frac{1}{J} = \nabla \xi^1 \cdot (\nabla \xi^2 \times \nabla \xi^3) \Leftarrow x_{\xi^i} = J(\nabla \xi^l \times \nabla \xi^m)$$

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let $b = \bar{b}^1 x_{\xi^1} + \dots + \bar{b}^n x_{\xi^n}$

Cartesian $\rightarrow$ $b \cdot \nabla\xi^i = \bar{b}^i$ ← ~~curvilinear~~

$$b = \bar{b}_1 \nabla\xi^1 + \dots + \bar{b}_n \nabla\xi^n$$

$$b \cdot x_{\xi^i} = \bar{b}_i$$

Metric Tensors

$$g_{ij} = \frac{\partial x}{\partial \xi^i} \cdot \frac{\partial x}{\partial \xi^j} \qquad g_{ij} g^{jk} = \delta_i^k$$

$$g = jj^T \qquad \det(g) = J^2$$

$$j^{-1} = \frac{1}{J} \begin{pmatrix} j'_{11} & & \\ & \ddots & \\ & & j'_{nn} \end{pmatrix}^T \qquad j'_{ij} \text{ is cofactor (adjoint) of } j_{ij}$$

$$\therefore \frac{\partial \xi^i}{\partial x^j} = \frac{1}{J}\left(\frac{\partial x^{j+1}}{\partial \xi^{i+1}} \frac{\partial x^{j+2}}{\partial \xi^{i+2}} - \frac{\partial x^{j+1}}{\partial \xi^{i+2}} \frac{\partial x^{j+2}}{\partial \xi^{i+1}} \right)$$

for 3D

$$\frac{\partial x^i}{\partial \xi^j} = J\left(\frac{\partial \xi^{j+1}}{\partial x^{i+1}} \frac{\partial \xi^{j+2}}{\partial x^{i+2}} - \frac{\partial \xi^{j+1}}{\partial x^{i+2}} \frac{\partial \xi^{j+2}}{\partial x^{i+1}} \right)$$

$$\frac{\partial \xi^i}{\partial x^j} = \frac{1}{J} (-1)^{i+j} \frac{\partial x^{3-j}}{\partial \xi^{3-i}}$$

for 2D

$$\frac{\partial x^i}{\partial \xi^j} = J (-1)^{i+j} \frac{\partial \xi^{3-j}}{\partial x^{3-i}}$$

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Christoffel Symbols

$$x_{\xi^i\xi^j} = \Gamma^k_{ij}\, x_{\xi^k}$$

$$\Downarrow$$

$$\Gamma^k_{ij} = x_{\xi^i\xi^j} \cdot \nabla \xi^k = \frac{\partial^2 x^l}{\partial\xi^i\partial\xi^j} \cdot \frac{\partial \xi^k}{\partial x^l}$$

$$x_{\xi^i\xi^j} = [ij,k]\, \nabla\xi^k$$

$$\therefore [ij,k] = x_{\xi^i\xi^j} \cdot x_{\xi^k} = \frac{\partial^2 x^l}{\partial\xi^i\partial\xi^j} \cdot \frac{\partial x^l}{\partial \xi^k}$$

$$[ij,k]\, g^{mk} = \frac{\partial^2 x^l}{\partial\xi^i\partial\xi^j} \cdot \frac{\partial x^l}{\partial\xi^k}\frac{\partial\xi^m}{\partial x^p}\frac{\partial\xi^k}{\partial x^p}$$

$$= \frac{\partial^2 x^l}{\partial\xi^i\partial\xi^j}\frac{\partial\xi^m}{\partial x^l}$$

$$= \Gamma^m_{ij}$$

$$[ij,k] = \frac{\partial}{\partial\xi^i}\left(\frac{\partial x^l}{\partial\xi^j}\frac{\partial x^l}{\partial\xi^k}\right) - \frac{\partial x^l}{\partial\xi^j}\frac{\partial}{\partial\xi^i}\left(\frac{\partial x^l}{\partial\xi^k}\right)$$

$$= \frac{\partial}{\partial\xi^i}\left(g_{jk}\right) - \frac{\partial x^l}{\partial\xi^j}\frac{\partial}{\partial\xi^i}\left(\frac{\partial x^l}{\partial\xi^k}\right)$$

$$[ij,k] = \frac{\partial}{\partial\xi^j}\left(g_{ik}\right) - \frac{\partial x^l}{\partial\xi^i}\frac{\partial}{\partial\xi^j}\left(\frac{\partial x^l}{\partial\xi^k}\right) \quad \to \frac{\partial}{\partial\xi^k}\left(\frac{\partial x^l}{\partial\xi^i}\frac{\partial x^l}{\partial\xi^j}\right)$$

$$\therefore [ij,k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial\xi^j} + \frac{\partial g_{jk}}{\partial\xi^i} - \frac{\partial g_{ij}}{\partial\xi^k}\right)$$

$$\Gamma^k_{ij} = \frac{1}{2} g^{km}\left(\frac{\partial g_{im}}{\partial\xi^j} + \frac{\partial g_{jm}}{\partial\xi^i} - \frac{\partial g_{ij}}{\partial\xi^m}\right)$$

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Jacobian

$A = (a_{ij})$, $i, j = 1, \dots, n$, $a_{ij} = a_{ij}(x)$

$\det(A) = \sum_j a_{ij} G_{ij}$, G is cofactor

$= \epsilon_{i_1 i_2 \dots i_n} a_{i_1} a_{2 i_2} \cdots a_{n i_n}$

$\frac{\partial}{\partial x} \det(A) = \frac{\partial a_{ij}}{\partial x} G_{ij}$

$J = \frac{\partial x^1}{\partial \xi^i} G^{11} + \frac{\partial x^2}{\partial \xi^i} G^{21} + \dots + \frac{\partial x^n}{\partial \xi^i} G^{n1} \quad \times \frac{\partial \xi^1}{\partial x^i}$

$0 = \frac{\partial x^1}{\partial \xi^2} G^{11} + \frac{\partial x^2}{\partial \xi^2} G^{21} + \dots + \frac{\partial x^n}{\partial \xi^2} G^{n1} \quad \times \frac{\partial \xi^2}{\partial x^i}$

$\vdots$

(summation)

$\frac{\partial \xi^1}{\partial x^i} J = G^{i1}$

$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$, $z(x, y)$ | $\frac{d}{dz}\left(\frac{dz}{dx}\right) \neq 0$, $= \frac{1}{\partial z}$ $z = x^i$ $z \neq 0$

$\therefore G^{im} = J \frac{\partial \xi^m}{\partial x^i}$

$\therefore \frac{\partial J}{\partial \xi^k} = J \frac{\partial^2 x^l}{\partial \xi^k \partial \xi^m} \frac{\partial \xi^m}{\partial x^l} = J \frac{\partial}{\partial x^l}\left(\frac{\partial x^l}{\partial \xi^k}\right) = J\, D \cdot \frac{\partial x}{\partial \xi^k}$

$\frac{\partial}{\partial \xi^j}\left(J \frac{\partial \xi^j}{\partial x^i}\right) = \frac{\partial J}{\partial \xi^j} \frac{\partial \xi^j}{\partial x^i} + J \frac{\partial^2 \xi^j}{\partial x^k \partial x^i} \frac{\partial x^k}{\partial \xi^j}$

$= J \frac{\partial^2 x^l}{\partial \xi^j \partial \xi^m} \frac{\partial \xi^m}{\partial x^l} \frac{\partial \xi^j}{\partial x^i} + J \left[\frac{\partial}{\partial x^i}\left(\frac{\partial \xi^j}{\partial x^k} \frac{\partial x^k}{\partial \xi^j}\right) - \frac{\partial \xi^j}{\partial x^k} \frac{\partial^2 x^k}{\partial \xi^j \partial \xi^m} \frac{\partial \xi^m}{\partial x^i} \right]$

$= 0$

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Conservation Law:

$$\nabla\cdot A = 0 \quad \frac{\partial A^i}{\partial x^i} = 0 \Rightarrow \frac{\partial A^i}{\partial \xi^j}\frac{\partial \xi^j}{\partial x^i} = 0$$

$$\because \frac{\partial}{\partial \xi^i}\left(J\frac{\partial \xi^i}{\partial x^j}\right) = 0$$

$$\because \bar{A}^j = A^i\frac{\partial \xi^j}{\partial x^i} \quad (\text{recall that } A = \bar{A}^1 x_{\xi_1} + \dots + \bar{A}^n x_{\xi_n})$$

$$\therefore A^i = \bar{A}^j\frac{\partial x^i}{\partial \xi^j}$$

$$\therefore A^i\frac{\partial}{\partial \xi^j}\left(J\frac{\partial \xi^j}{\partial x^i}\right) + J\frac{\partial A^i}{\partial \xi^j}\frac{\partial \xi^j}{\partial x^i} = 0$$

$$\Downarrow$$

$$\frac{\partial}{\partial \xi^j}\left(J\bar{A}^j\right) = 0$$

if $\frac{\partial A^i}{\partial x^i} = F$, then $\frac{1}{J}\frac{\partial}{\partial \xi^j}\left(J\bar{A}^j\right) = F$

$$\frac{\partial A^{ij}}{\partial x^j} = F^i$$

assume A^{ij} is a tensor, so $\bar{A}^{ij} = A^{mn}\frac{\partial \xi^i}{\partial x^m}\frac{\partial \xi^j}{\partial x^n}$

$$\therefore \frac{\partial}{\partial x^j}\left(\bar{A}^{mn}\frac{\partial x^i}{\partial \xi^m}\frac{\partial x^j}{\partial \xi^n}\right)$$

$$= \frac{\partial \bar{A}^{mn}}{\partial \xi^n}\frac{\partial x^i}{\partial \xi^m} + \bar{A}^{mn}\frac{\partial^2 x^i}{\partial \xi^m \partial \xi^n} + \bar{A}^{mn}\frac{\partial x^i}{\partial \xi^m}\frac{\partial^2 x^j}{\partial \xi^n \partial \xi^l}\frac{\partial \xi^l}{\partial x^j}$$

$$= \frac{\partial \bar{A}^{mn}}{\partial \xi^n}\frac{\partial x^i}{\partial \xi^m} + \bar{A}^{mn}\frac{\partial x^i}{\partial \xi^m \partial \xi^n} + \frac{1}{J}\bar{A}^{mn}\frac{\partial J}{\partial \xi^n}\frac{\partial x^i}{\partial \xi^m} = F^i$$

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$\cdots + \bar{A}^{n} x_{\xi^n})$

$\dfrac{\partial \xi^j}{\partial x^n}$

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$$\therefore \frac{\partial \bar{A}^{pn}}{\partial \xi^n} + \bar{A}^{mn} \frac{\partial^2 x^i}{\partial \xi^m \partial \xi^n} \frac{\partial \xi^p}{\partial x^i} + \frac{1}{J} \bar{A}^{pn} \frac{\partial J}{\partial \xi^n} = F^i \frac{\partial \xi^p}{\partial x^i} = \bar{F}^p$$

$$\frac{1}{J} \frac{\partial}{\partial \xi^n} \left(\bar{A}^{pn} J \right) + \Gamma^p_{mn} \bar{A}^{mn} = \bar{F}^p$$

Time - dependent

let $\xi^0 = \tau$

$x^0 = t$

$t = \tau$

flow velocity $\quad w = \bar{w}^i x_{\xi^i}$

$$\bar{w}^i = w \cdot \nabla \xi^i = w^j \frac{\partial \xi^i}{\partial x^j} = \frac{\partial x^j}{\partial \tau} \frac{\partial \xi^i}{\partial x^j}$$

$$w^i = \bar{w}^j \frac{\partial x^i}{\partial \xi^j}$$

$$\because \frac{\partial \xi^i}{\partial \xi^0} = 0$$

$$\therefore \frac{\partial \xi^i}{\partial x^0} \frac{\partial x^0}{\partial \xi^0} + \frac{\partial \xi^i}{\partial x^j} \frac{\partial x^j}{\partial \xi^0} = 0$$

$$\therefore \frac{\partial \xi^i}{\partial t} = - \frac{\partial x^j}{\partial \tau} \frac{\partial \xi^i}{\partial x^j} = - \bar{w}^i$$

$$\frac{1}{J} \frac{\partial}{\partial \tau}(J) = \frac{\partial^2 x^k}{\partial \xi^m \partial \tau} \frac{\partial \xi^m}{\partial x^k} = \frac{\partial}{\partial \xi^m} \left(\bar{w}^j \frac{\partial x^k}{\partial \xi^j} \right) \frac{\partial \xi^m}{\partial x^k}$$

$$= \frac{\partial \bar{w}^m}{\partial \xi^m} + \frac{1}{J} \bar{w}^j \frac{\partial J}{\partial \xi^j} \qquad j = 1, \ldots, n$$

$$= \frac{1}{J} \frac{\partial}{\partial \xi^j} \left(J \bar{w}^j \right)$$

Geometry (differential)

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$$d\vec{r} = \frac{d\vec{r}}{d\xi^i} d\xi^i$$

let e_{ξ^i} be unit vector along $\frac{d\vec{r}}{d\xi^i}$

$$\therefore d\vec{r} = \left|\frac{d\vec{r}}{d\xi^i}\right| d\xi^i e_{\xi^i} = h_{\xi^i} d\xi^i e_{\xi^i}$$

$$dl = \sqrt{d\vec{r}^2} = \sqrt{h_{\xi^i}^2 d\xi^{i^2}}$$

---

for a scaler function f,

$$df = \nabla f \cdot dr$$

$$= \frac{\partial f}{\partial \xi^i} d\xi^i$$

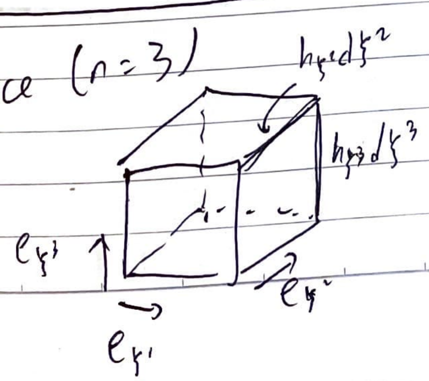
$$= \frac{\partial f}{\partial \xi^i} \left(\frac{1}{h_{\xi^i}} d\vec{r} \cdot e_{\xi^i} \right)$$

$$= \left(\frac{1}{h_{\xi^i}} \frac{\partial f}{\partial \xi^i} e_{\xi^i} \right) \cdot d\vec{r}$$

$$= \nabla f \cdot d\vec{r}$$

$$\therefore \nabla f = \frac{1}{h_\xi} \frac{\partial f}{\partial \xi^i} e_{\xi^i}$$

Divergence ($n=3$)



for surfaces perpendicular to e_{ξ^1}

area $= h_{\xi^2} h_{\xi^3} d\xi^2 d\xi^3$

$\therefore \quad \frac{\partial f_1}{\partial \xi^1} d\xi^1 = \frac{\partial}{\partial \xi^1}\left(f_{\xi^1} h_{\xi^2} h_{\xi^3}\right) d\xi^1 d\xi^2 d\xi^3$

Similarly for e_{ξ^2}, e_{ξ^3}, so

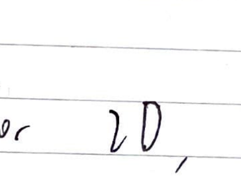
$$\nabla \cdot \vec{f} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint \vec{f} \cdot \hat{n}\, ds$$

$$= \lim_{\Delta V \to 0} \frac{1}{h_{\xi^1} h_{\xi^2} h_{\xi^3}} \left(\frac{\partial}{\partial \xi^1}\left(f_{\xi^1} h_{\xi^2} h_{\xi^3}\right) + \frac{\partial}{\partial \xi^2}\left(f_{\xi^2} h_{\xi^1} h_{\xi^3}\right) + \frac{\partial}{\partial \xi^3}\left(f_{\xi^3} h_{\xi^1} h_{\xi^2}\right) \right)$$

$$= \frac{1}{h_{\xi^1} h_{\xi^2} h_{\xi^3}} \left(\frac{\partial}{\partial \xi^1}\left(f_{\xi^1} h_{\xi^2} h_{\xi^3}\right) + \frac{\partial}{\partial \xi^2}\left(f_{\xi^2} h_{\xi^1} h_{\xi^3}\right) + \frac{\partial}{\partial \xi^3}\left(f_{\xi^3} h_{\xi^1} h_{\xi^2}\right) \right)$$

Curl $(n=3)$

$$\nabla \times \vec{f} = \lim_{\Delta A \to 0} \frac{1}{\Delta A} \oint \vec{f} \cdot d\vec{r}$$



along e_{ξ^1} : $f_{\xi^1} h_{\xi^1} d\xi^1 \to -\frac{\partial}{\partial \xi^2}\left(f_{\xi^1} h_{\xi^1}\right) d\xi^1 d\xi^2$

along e_{ξ^2} : $f_{\xi^2} h_{\xi^2} d\xi^2 \to \frac{\partial}{\partial \xi^1}\left(f_{\xi^2} h_{\xi^2}\right) d\xi^1 d\xi^2$

$\therefore$ for 2D, $\nabla \times \vec{f} = \frac{1}{h_{\xi^1} h_{\xi^2}} \left(\frac{\partial}{\partial \xi^1}\left(f_{\xi^2} h_{\xi^2}\right) - \frac{\partial}{\partial \xi^2}\left(f_{\xi^1} h_{\xi^1}\right) \right)$

$\Big[f_{\xi^1} h_{\xi^1}\left(\xi^2 - \frac{d\xi^2}{2}\right) d\xi^1 - f_{\xi^1} h_{\xi^1}\left(\xi^2 + \frac{d\xi^2}{2}\right) d\xi^1$

$= -\frac{\partial}{\partial \xi^2}\left(f_{\xi^1} h_{\xi^1}\right) d\xi^1 d\xi^2 \Big]$

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Geometry (differential)

for 3D, $\nabla \times \vec{f} = \dfrac{1}{h_{\xi_1} h_{\xi_2} h_{\xi_3}} \begin{vmatrix} h_{\xi_1} e_{\xi_1} & h_{\xi_2} e_{\xi_2} & h_{\xi_3} e_{\xi_3} \\ \frac{\partial}{\partial \xi_1} & \frac{\partial}{\partial \xi_2} & \frac{\partial}{\partial \xi_3} \\ f_{\xi_1} h_{\xi_1} & f_{\xi_2} h_{\xi_2} & f_{\xi_3} h_{\xi_3} \end{vmatrix}$

Cylindrical Coordinates :

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$h_r = \sqrt{\left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2} = 1$

$h_\theta = r$

$h_z = 1$

$$\therefore \nabla^2 f = \frac{1}{h_r h_\theta h_z}\left[\frac{\partial}{\partial r}\left(h_\theta h_z \frac{1}{h_r}\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(h_r h_z \frac{1}{h_\theta}\frac{\partial f}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(h_r h_\theta \frac{1}{h_z}\frac{\partial f}{\partial z}\right)\right]$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial f}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial f}{\partial z}\right)\right]$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial f}{\partial z^2}$$

Spherical Coordinates :

$x = r\cos\phi\sin\theta$

$y = r\sin\phi\sin\theta$

$z = r\cos\theta$

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Geometry (differential)

$h_r = 1$

$h_\theta = \sqrt{r^2\cos^2\phi\cos^2\theta + r^2\sin^2\phi\cos^2\theta + r^2\sin^2\theta} = r$

$h_\phi = \sqrt{r^2\sin^2\phi\sin^2\theta + r^2\cos^2\phi\sin^2\theta + 0} = r\sin\theta$

$$\therefore \nabla^2 f = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial f}{\partial\phi}\right)\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

(note)

$$\frac{\partial A^i}{\partial x^i} = \frac{1}{J}\left[J\frac{\partial A^i}{\partial\xi^j}\frac{\partial\xi^j}{\partial x^i} + A^i\frac{\partial}{\partial\xi^j}\left(J\frac{\partial\xi^j}{\partial x^i}\right)\right]$$

$$= \frac{1}{J}\left[\frac{\partial}{\partial\xi^j}\left(JA^i\frac{\partial\xi^j}{\partial x^i}\right)\right]$$

$$= \frac{1}{J}\frac{\partial}{\partial\xi^j}\left(J\bar{A}^j\right) = \frac{1}{J}\frac{\partial}{\partial\xi^j}\left(A^i G^{ij}\right)$$

$A = \bar{A}^i x_{\xi^i} = A_{(i)}e_{\xi^i}$

$\therefore A_{(i)} = h_i A^i$

$A^i = \dfrac{A_{(i)}}{h_{\xi^i}}$

Unit vector (cylindrical coordinates)

$$e_{\xi^i} = \frac{\partial\vec{r}}{\partial\xi^i}\Big/\left|\frac{\partial\vec{r}}{\partial\xi^i}\right| = \frac{1}{\left|\frac{\partial\vec{r}}{\partial\xi^i}\right|}\left(\frac{\partial x^1}{\partial\xi^i}e_1 + \dots + \frac{\partial x^n}{\partial\xi^i}e_n\right)$$

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

Geometry (differential)

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$e_r = \cos\theta\, e_1 + \sin\theta\, e_2$

$e_\theta = (-r\sin\theta\, e_1 + r\cos\theta\, e_2)\frac{1}{r}$

$e_z = e_3$

$\frac{\partial e_r}{\partial r} = 0 \qquad \frac{\partial e_r}{\partial \theta} = -\sin\theta\, e_1 + \cos\theta\, e_2 = e_\theta \qquad \frac{\partial e_r}{\partial z} = 0$

$\frac{\partial e_\theta}{\partial r} = 0 \qquad \frac{\partial e_\theta}{\partial \theta} = -e_r \qquad \frac{\partial e_\theta}{\partial z} = 0$

$\frac{\partial e_z}{\partial r} = 0 \qquad \frac{\partial e_z}{\partial \theta} = 0 \qquad \frac{\partial e_z}{\partial z} = 1$

Unit vector (Spherical coordinates)

$x = r\cos\phi\sin\theta$

$y = r\sin\phi\sin\theta$

$z = r\cos\theta$

$e_r = \cos\phi\sin\theta\, e_1 + \sin\phi\sin\theta\, e_2 + \cos\theta\, e_3$

$e_\theta = (r\cos\phi\cos\theta\, e_1 + r\sin\phi\cos\theta\, e_2 - r\sin\theta\, e_3)\frac{1}{r}$

$e_\phi = (-r\sin\phi\sin\theta\, e_1 + r\cos\phi\sin\theta\, e_2)\frac{1}{r\sin\theta}$

$\frac{\partial e_r}{\partial r} = 0 \qquad \frac{\partial e_r}{\partial \theta} = e_\theta \qquad \frac{\partial e_r}{\partial \phi} = \sin\theta\, e_\phi \qquad \frac{\partial e_\phi}{\partial r} = 0 \qquad \frac{\partial e_\phi}{\partial \theta} = 0$

$\frac{\partial e_\theta}{\partial r} = 0 \qquad \frac{\partial e_\theta}{\partial \theta} = -e_r \qquad \frac{\partial e_\theta}{\partial \phi} = \cos\theta\, e_\phi \qquad \frac{\partial e_\phi}{\partial \phi} = -\cos\phi\, e_1 - \sin\phi\, e_2 = -\sin\theta\, e_r - \cos\theta\, e_\theta$

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Date. 13 · 10 · 2017 Geometry (differential)

$$\cancel{e_i \frac{\partial A}{\partial x^i} = e_i \frac{\partial A}{\partial \xi^j}\frac{\partial \xi^j}{\partial x^i}}$$

$$\cancel{= x_{\xi e} \frac{\partial \xi^e}{\partial x^i}\frac{\partial \xi^j}{\partial x^i}\frac{\partial A}{\partial \xi^j}}$$

$$\cancel{= x_{\xi e}\, g^{ej} \frac{\partial A}{\partial \xi^j}}$$

$$\cancel{= h_{\xi e}\, g^{ej} \frac{\partial A}{\partial \xi^j} \hat{x}_{\xi e}}$$

~~if orthogonal,~~ $\cancel{g^{ij} = g^{ij}\delta^i_j}$, $\cancel{g^{ii} = \frac{1}{g_{ii}} = \frac{1}{h_{\xi i}^2}}$

$$\cancel{\therefore e_i \frac{\partial A}{\partial x^i} = \frac{1}{h_{\xi e}} \frac{\partial A}{\partial \xi^e} \hat{x}_{\xi e}}$$

∇A

let C be a curve, tangent is given by:

$$\tau = \frac{dx}{ds} = \frac{dx}{\partial \xi^i}\frac{\partial \xi^i}{\partial s} = \frac{\partial \xi^j}{\partial s} x_{\xi j} = \tau^j x_{\xi j}$$

$$\nabla A \cdot \tau = \frac{dA}{ds} = \left(\frac{\partial A}{\partial \xi^i}\frac{\partial \xi^i}{\partial x}\right) \cdot \left(\tau^j x_{\xi j}\right)$$

$$= \left(\frac{\partial A}{\partial \xi^i} x^{\xi i}\right) \cdot \left(\tau^j x_{\xi j}\right)$$

recall that $x_{\xi i} \cdot x^{\xi j} = \delta^j_i$

$\therefore$ we can define ∇ as $x^{\xi i} \frac{\partial}{\partial \xi^i}$

Geometry (differential)

$\nabla \vec{A}$

$$\cancel{\nabla\vec{A} = \nabla_j A_i = x^{\xi j}\frac{\partial A_i}{\partial \xi^j}}$$

$$\nabla\vec{A} = x^{\xi j}\frac{\partial \vec{A}}{\partial\xi^j} = x^{\xi j}\frac{\partial}{\partial \xi^j}(A^i x_{\xi i})$$

$$= x^{\xi j}\left[\frac{\partial A^i}{\partial \xi^j}x_{\xi i} + A^i x_{\xi i \xi j}\right]$$

$$= x^{\xi j}\left[\frac{\partial A^i}{\partial\xi^j}x_{\xi i} + A^i\Gamma^k_{ij}x_{\xi k}\right]$$

$$= x^{\xi j}A^i|_j x_{\xi i}$$

$$= A^i|_j x^{\xi j}x_{\xi i}$$

$$\because x^{\xi i}{}_{\xi j} = \frac{\partial}{\partial \xi^j}\left(\frac{\partial \xi^i}{\partial x}\right) = \frac{\partial^2\xi^i}{\partial x^k\partial x}\frac{\partial x^k}{\partial \xi^j} = -\frac{\partial\xi^i}{\partial x^k}\frac{\partial^2 x^k}{\partial\xi^m\partial\xi^j}\frac{\partial\xi^m}{\partial x} = -\Gamma^i_{mj}x^{\xi m} = -\Gamma^i_{jm}x^{\xi m}$$

$$\therefore \nabla\vec{A} = x^{\xi j}\frac{\partial}{\partial\xi^j}(A_i x^{\xi i})$$

$$= x^{\xi j}\left(\frac{\partial A_i}{\partial\xi^j}x^{\xi i} + A_i x^{\xi i}{}_{\xi j}\right)$$

$$= x^{\xi j}\left(\frac{\partial A_i}{\partial \xi^j}x^{\xi i} - A_i\Gamma^i_{jk}x^{\xi k}\right)$$

$$= A_i|_j x^{\xi j}x^{\xi i}$$

$\nabla^2\vec{A}$

$$\nabla^2\vec{A} = x^{\xi k}\frac{\partial}{\partial \xi^k}(A^{ij}x_{\xi i}x_{\xi j}) = x^{\xi k}\left[\frac{\partial A^{ij}}{\partial\xi^k}x_{\xi i}x_{\xi j} + A^{ij}\Gamma^m_{ik}x_{\xi m}x_{\xi j} + A^{ij}\Gamma^n_{kj}x_{\xi i}x_{\xi n}\right]$$

$$= \left(\frac{\partial A^{ij}}{\partial\xi^k} + A^{mj}\Gamma^i_{mk} + A^{in}\Gamma^j_{kn}\right)x_{\xi i}x_{\xi j}x^{\xi k}$$

$$= A^{ij}|_k \, x_{\xi^i} x_{\xi^j} x^{\xi^k}$$

$\nabla \cdot \vec{A}$

$$\nabla \cdot \vec{A} = \nabla\vec{A} : I = A^i|_i \, x^{\xi^i} x_{\xi^i}$$

$\nabla \cdot \vec{\vec{A}}$

$$\nabla \cdot \vec{\vec{A}} = \nabla\vec{\vec{A}} : I = A^{ij}|_i \, x_{\xi^i} x_{\xi^j} x^{\xi^i}$$

$\nabla \times \vec{A}$

$$\nabla \times \vec{A} = x^{\xi^j} \times \frac{\partial \vec{A}}{\partial \xi^j} = x^{\xi^j} \times \frac{\partial}{\partial \xi^j}\left(A^i x_{\xi^i}\right)$$

$$= x^{\xi^j} \times x_{\xi^i} \frac{\partial A^i}{\partial \xi^j} + A^i x^{\xi^j} x_{\xi^i \xi^j}$$

$$= \frac{\partial A^i}{\partial \xi^j} x^{\xi^j} \times x_{\xi^i} + A^i \Gamma^k_{ij} x_{\xi^k}$$

$$= A^i|_j \, x^{\xi^j} \times x_{\xi^i}$$ ← note that $\frac{\partial A^i}{\partial \xi^j} - \frac{\partial A^j}{\partial \xi^i}$ is not a tensor

$$\nabla \times \vec{A} = x^{\xi^j} \times \frac{\partial}{\partial \xi^j}\left(A_i x^{\xi^i}\right) = x^{\xi^j} \times x^{\xi^i}\left(\frac{\partial A_i}{\partial \xi^j} - A_k \Gamma^k_{ij}\right)$$

note that, for example, $(\nabla \times \vec{A})_\phi = x^r \times x^\theta \left(\underbrace{\frac{\partial A_\theta}{\partial r} - \frac{\partial A_r}{\partial \theta}} + A_k \Gamma^k_{r\theta} - A_k \Gamma^k_{\theta r}\right)$

a tensor, however its not the usual curl because it had two indices.

$$\therefore \boxed{\nabla \times \vec{A} \equiv \epsilon^{ijk} \partial_i A_j = \frac{\varepsilon^{ijk}}{\sqrt{g}} \partial_i A_j}$$

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Geometry (differential)

$$x_{\xi^i} = \frac{\partial \vec{x}}{\partial \xi^i}, \quad g^{ij} g_{jk} = \frac{\partial \xi^i}{\partial x^m}\frac{\partial \xi^j}{\partial x^m}\frac{\partial x^n}{\partial \xi^j}\frac{\partial x^n}{\partial \xi^k} = \frac{\partial \xi^i}{\partial x^m}\frac{\partial x^m}{\partial \xi^k} = \delta^i_k$$

$$x_{\xi^i} g^{ij} = \frac{\partial x^m}{\partial \xi^i} \cdot \frac{\partial \xi^i}{\partial x^n}\frac{\partial \xi^j}{\partial x^n} = \delta_{mn}\frac{\partial \xi^j}{\partial x^n} = \frac{\partial \xi^j}{\partial x^m} = x^{\xi^j}$$

$$x^{\xi^i} g_{ij} = \frac{\partial \xi^i}{\partial x^m}\frac{\partial x^n}{\partial \xi^i}\frac{\partial x^n}{\partial \xi^j} = \delta_{mn}\frac{\partial x^n}{\partial \xi^j} = x_{\xi^j}$$

physical quantity $A_{(i)}$, $A = A_{(i)} e_{(i)}$

$$A = A^i x_{\xi^i} = A_i x^{\xi^i}$$

$$= A^i h_{\xi^i} e_{\xi^i} = A_i \frac{1}{h_{\xi^i}} e^{\xi^i} = A_{(i)} e_{(i)}$$

$$\therefore A^i = \frac{A_{(i)}}{h_{\xi^i}} \qquad A_i = h_{\xi^i} A_{(i)}$$

Orthogonal coordinates

$$x_{\xi^i} \cdot x_{\xi^j} = h_{ij}^2 \delta_{ij} \implies g_{ij} = g^{ij} = 0 \quad \text{if } i \neq j$$

$$\cancel{g_{ii}\Gamma^i_{jk} = g_{ii}\frac{\partial x_{\xi^j}}{\partial \xi^k} \cdot x^{\xi^i} = \frac{\partial x_{\xi^j}}{\partial \xi^k} \cdot x_{\xi^i}}$$

$$\Gamma^i_{jk} = \frac{1}{2} g^{im}\left(\frac{\partial g_{jm}}{\partial \xi^k} + \frac{\partial g_{km}}{\partial \xi^j} - \frac{\partial g_{jk}}{\partial \xi^m}\right)$$

if $i \neq j \neq k$, each term is obviously zero

$\therefore \Gamma^i_{jk} = 0$ when $i \neq j \neq k$

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from property of Christoffel's symbol,

$$\Gamma^i_{jk} = \Gamma^i_{kj}$$

$$\Gamma^i_{ij} = \Gamma^i_{ji} = \frac{1}{2} g^{im}\left(\frac{\partial g_{im}}{\partial \xi^j} + \frac{\partial g_{jm}}{\partial \xi^i} - \frac{\partial g_{ij}}{\partial \xi^m}\right) \qquad i \neq j$$

$$= \frac{1}{2} g^{ii}\left(\frac{\partial g_{ii}}{\partial \xi^j} + \frac{\partial g_{ji}}{\partial \xi^i}\right)$$

$$= \frac{1}{2} g^{ii} \frac{\partial g_{ii}}{\partial \xi^j}$$

$$\Gamma^i_{jj} = -\frac{1}{2} g^{ii} \frac{\partial g_{jj}}{\partial \xi^i} \quad \cancel{= \frac{1}{4} \frac{\partial}{\partial \xi^i}(g^{ii})^2} \qquad \therefore \Gamma^i_{ii} = \frac{1}{2} g^{ii} \frac{\partial g_{ii}}{\partial \xi^i}$$

Cylindrical coordinates

$x = r\cos\theta$

$y = r\sin\theta \qquad (x, y, z) \rightarrow (r, \theta, z)$

$z = z$

$$g_{rr} = \frac{\partial \vec{x}}{\partial r} \cdot \frac{\partial \vec{x}}{\partial r} = \cos^2\theta + \sin^2\theta = 1$$ [see Unit Vector (4/10/2017)]

$$g_{\theta\theta} = \frac{\partial \vec{x}}{\partial \theta} \cdot \frac{\partial \vec{x}}{\partial \theta} = r^2(\cos^2\theta + \sin^2\theta) = r^2$$

$$g_{zz} = 1$$

$$\Gamma^r_{rr} = \frac{1}{2} g^{rr} \frac{\partial g_{rr}}{\partial r} = 0 \qquad \Gamma^r_{r\theta} = \Gamma^r_{\theta r} = \Gamma^r_{rz} = \Gamma^r_{zr} = 0$$

$$\Gamma^\theta_{\theta\theta} = \frac{1}{2} g^{\theta\theta} \frac{\partial g_{\theta\theta}}{\partial \theta} = 0 \qquad \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{2} g^{\theta\theta} \frac{\partial g_{\theta\theta}}{\partial r} = \frac{1}{2r^2} \cdot 2r = \frac{1}{r}$$

$$\Gamma^z_{zz} = 0 \qquad \Gamma^\theta_{z\theta} = \Gamma^\theta_{\theta z} = 0$$

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$$\Gamma^z_{rz} = \Gamma^z_{zr} = \Gamma^z_{\theta z} = \Gamma^z_{z\theta} = 0$$

$$\Gamma^r_{\theta\theta} = -\frac{1}{2} g^{rr} \frac{\partial g_{\theta\theta}}{\partial r} = -r \qquad \Gamma^r_{zz} = 0$$

$$\Gamma^\theta_{rr} = \Gamma^\theta_{zz} = 0$$

$$\Gamma^z_{rr} = 0 \qquad \Gamma^z_{\theta\theta} = -\frac{1}{2} g^{zz} \frac{\partial g_{\theta\theta}}{\partial z} = 0$$

Spherical coordinates

$x = r\cos\phi\sin\theta$

$y = r\sin\phi\sin\theta$

$z = r\cos\theta$

$$g_{rr} = \frac{\partial \vec{x}}{\partial r}\cdot\frac{\partial \vec{x}}{\partial r} = 1 \qquad \text{[see Unit Vector 4/10/2017]}$$

$$g_{\theta\theta} = \frac{\partial \vec{x}}{\partial \theta}\cdot\frac{\partial \vec{x}}{\partial \theta} = r^2$$

$$g_{\phi\phi} = \frac{\partial \vec{x}}{\partial \phi}\cdot\frac{\partial \vec{x}}{\partial \phi} = r^2\sin^2\theta$$

$$\Gamma^r_{rr} = 0 \qquad \Gamma^r_{r\theta} = \Gamma^r_{\theta r} = \Gamma^r_{r\phi} = \Gamma^r_{\phi r} = 0$$

$$\Gamma^\theta_{\theta\theta} = 0 \qquad \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{2} g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial r} = \frac{1}{2r^2}\cdot 2r = \frac{1}{r}$$

$$\Gamma^\phi_{\phi\phi} = 0 \qquad \Gamma^\theta_{\phi\theta} = \Gamma^\theta_{\theta\phi} = 0$$

$$\Gamma^r_{\theta\theta} = -\frac{1}{2} g^{rr}\frac{\partial g_{\theta\theta}}{\partial r} = -r \qquad \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{2} g^{\phi\phi}\frac{\partial g_{\phi\phi}}{\partial r} = \frac{1}{2r^2\sin^2\theta}\cdot 2r\sin^2\theta = \frac{1}{r}$$

$$\Gamma^r_{\phi\phi} = -\frac{1}{2} g^{rr}\frac{\partial g_{\phi\phi}}{\partial r} = -r\sin^2\theta \qquad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \frac{1}{2} g^{\phi\phi}\frac{\partial g_{\phi\phi}}{\partial \theta} = \frac{1}{2r^2\sin^2\theta}\cdot 2r^2\sin\theta\cos\theta$$

$$= \cot\theta$$

$$\Gamma^\theta_{rr} = 0$$

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$$\Gamma^{\theta}_{\phi\phi} = -\frac{1}{2} g^{\theta\theta}\frac{\partial g_{\phi\phi}}{\partial\theta} = -\frac{1}{2r^2}\cdot 2r^2\sin\theta\cos\theta = -\frac{\sin 2\theta}{2}$$

$$\Gamma^{\phi}_{rr} = 0$$

$$\Gamma^{\phi}_{\theta\theta} = -\frac{1}{2} g^{\phi\phi}\frac{\partial g_{\theta\theta}}{\partial\phi} = 0$$

Examples : (Spherical coordinates) $\left[v^r = v_r,\ v^\theta = \frac{v_\theta}{r},\ v^\phi = \frac{v_\phi}{r\sin\theta}\right]$

$$\nabla\cdot\vec{v} = x^{\xi^j}\frac{\partial}{\partial\xi^j}\left(v^i x_{\xi^i}\right) = A^i|_j\, x^{\xi^j}\cdot x_{\xi^i} = A^i|_i = \frac{\partial A^i}{\partial \xi^i} + A^j\Gamma^i_{ij}$$

$$= \frac{\partial v^r}{\partial r} + \frac{\partial v^\theta}{\partial\theta} + \frac{\partial v^\phi}{\partial\phi} + v^r\cdot\frac{1}{r} + v^r\frac{1}{r} + v^\theta\cot\theta$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, v_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

$$\vec{v}\cdot\nabla\vec{v} = v^i x_{\xi^i}\, x^{\xi^j}\frac{\partial}{\partial\xi^j}\left(v^j x_{\xi^j}\right) = A^i A^j|_i\, x^{\xi^i}\cdot x_{\xi^k}\, x_{\xi^j}$$

$$= v^i\left(\frac{\partial v^j}{\partial\xi^i} + v^k\Gamma^j_{ik}\right)x_{\xi^j}$$

if $j=1$,

$$(\vec{v}\cdot\nabla\vec{v})_r = v^r\frac{\partial v^r}{\partial r} + v^\theta\frac{\partial v^r}{\partial\theta} + v^\phi\frac{\partial v^r}{\partial\phi} + v^{\theta 2}(-r) + v^{\phi 2}(-r\sin^2\theta)$$

$$= v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{v_\theta^2 + v_\phi^2}{r}$$

if $j=2$

$$(\vec{v}\cdot\nabla\vec{v})_\theta = \left[v^r\frac{\partial v^\theta}{\partial r} + v^\theta\frac{\partial v^\theta}{\partial\theta} + v^\phi\frac{\partial v^\theta}{\partial\phi} + 2v^r v^\theta\left(\frac{1}{r}\right) + v^{\phi 2}\left(-\sin\theta\cos\theta\right)\right]\cdot r$$

$$= \left[v_r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{v_\theta}{r^2}\frac{\partial v_\theta}{\partial\theta} + \frac{v_\phi}{r^2\sin\theta}\frac{\partial v_\theta}{\partial\phi} + \frac{2v_r v_\theta}{r^2} - \frac{v_\phi^2}{r^2}\cot\theta\right]\cdot r$$

$$= v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2}{r}\cot\theta$$

if $j=3$

$$(\vec{v}\cdot\nabla\vec{v})_\phi = \left[v^r\frac{\partial v^\phi}{\partial r} + v^\theta\frac{\partial v^\phi}{\partial\theta} + v^\phi\frac{\partial v^\phi}{\partial\phi} + 2v^r v^\phi\left(\frac{1}{r}\right) + 2v^\theta v^\phi\left(\cot\theta\right)\right] r\sin\theta$$

$=$

Geometry (differential) 19/10/2017

$$\nabla \cdot \overset{=}{\tau} = \nabla \cdot \left(\nabla \vec{v} + (\nabla \vec{v})^T \right)$$

$$\nabla \vec{v} = x^{\xi i} \frac{\partial}{\partial \xi j} (v^i x_{\xi i}) = \left(\frac{\partial A^i}{\partial \xi^j} + A^k \Gamma^i_{kj} \right) x^{\xi j} x_{\xi i} = A^i|_j x^{\xi j} x_{\xi i}$$

$$(\nabla \vec{v})_{rr} = \frac{\partial v^r}{\partial r} = \frac{\partial v_r}{\partial r}$$

$$(\nabla \vec{v})_{r\theta} = \left[\frac{\partial v^r}{\partial \theta} + v^\theta (-r) \right] \frac{1}{r} = \frac{1}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r}$$

$$(\nabla \vec{v})_{\theta r} = \left[\frac{\partial v^\theta}{\partial r} + v^\theta \frac{1}{r} \right] \cdot r = r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{v_\theta}{r}$$

$$(\nabla \vec{v})_{r\phi} = \left[\frac{\partial v^r}{\partial \phi} + v^\phi (-r \sin^2 \theta) \right] \frac{1}{r \sin\theta} = \frac{1}{r \sin\theta} \frac{\partial v_r}{\partial \phi} - \frac{v_\phi}{r}$$

$$(\nabla \vec{v})_{\phi r} = \left[\frac{\partial v^\phi}{\partial r} + v^\phi \left(\frac{1}{r} \right) \right] r \sin\theta = r \frac{\partial}{\partial r} \left(\frac{v_\phi}{r} \right) + \frac{v_\phi}{r}$$

$$(\nabla \vec{v})_{\theta\theta} = \left[\frac{\partial v^\theta}{\partial \theta} + v^r \frac{1}{r} \right] = \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}$$

$$(\nabla \vec{v})_{\theta\phi} = \left[\frac{\partial v^\theta}{\partial \phi} + v^\phi (-\sin\theta \cos\theta) \right] \frac{1}{r \sin\theta} r = \frac{1}{\sin\theta} \frac{\partial}{\partial \phi} \left(\frac{v_\theta}{r} \right) - \frac{v_\phi}{r} \cot\theta$$

$$(\nabla \vec{v})_{\phi\theta} = \left[\frac{\partial v^\phi}{\partial \theta} + v^\phi \cot\theta \right] \frac{1}{r} r \sin\theta = \frac{\sin\theta}{r} \frac{\partial}{\partial \theta} \left(\frac{v_\phi}{\sin\theta} \right) + \frac{v_\phi}{r} \cot\theta$$

$$(\nabla \vec{v})_{\phi\phi} = \left[\frac{\partial v^\phi}{\partial \phi} + v^r \left(\frac{1}{r} \right) + v^\theta \cot\theta \right] = \frac{1}{r \sin\theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta}{r} \cot\theta$$

$$\nabla \cdot \nabla \vec{v} = \left(\frac{\partial A^i|_j}{\partial \xi^j} + A^k|_j \Gamma^i_{jk} - A^i|_k \Gamma^k_{jj} \right) x^{\xi j} x^{\xi j} x_{\xi i}$$

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extra

$$\tilde{\epsilon}^{ijk} \frac{\partial x^p}{\partial \xi^i} \frac{\partial x^q}{\partial \xi^j} \frac{\partial x^r}{\partial \xi^k} = \frac{\epsilon^{ijk}}{\sqrt{g}} \frac{\partial x^p}{\partial \xi^i} \frac{\partial x^q}{\partial \xi^j} \frac{\partial x^r}{\partial \xi^k} = \frac{\epsilon^{pqr}}{\sqrt{g}} \sqrt{\frac{\tilde{g}}{g}} = \frac{\epsilon^{pqr}}{\sqrt{g}} = \epsilon^{pqr}$$

$$\because J = \sqrt{\frac{\tilde{g}}{g}} = \begin{vmatrix} \frac{\partial x^1}{\partial \xi^1} & \frac{\partial x^1}{\partial \xi^2} \\ \frac{\partial x^2}{\partial \xi^1} & \cdot \\ & \cdot \end{vmatrix}$$